STA447/2006 Midterm #2, March 19, 2020

(ONLINE; 3 hours; 8 questions; 2 pages; total points = 50)

INSTRUCTIONS:

• **Your solutions should be entered ONLINE, in the "Midterm #2" item within the "Quizzes" tab on the course's Quercus web page.** You may either fill in the individual textbox for each question (1–8), or upload a separate file for each question (9–16), or upload ALL your answers as a single combined file in the final question (17).

• **Your solutions are due by 9:00 PM sharp** – be sure to submit them by then!

• If you are completely unable to connect to Quercus, then as a **last resort only**, you may instead email your solutions directly to: ali.al.aradi@utoronto.ca

• **This exam is open book, i.e. you may look at your textbook and notes.**

• **You may use results from class or the textbook**, with explanation.

• However, you **may not communicate** in any way with anyone else (besides the instructor or TA) during the test. All modes of communication with others are **strictly forbidden**, including in-person, phone, text, chat, email, discussion board, etc. Any such communication would **seriously violate** the university's Code of Behaviour on Academic Matters.

• **Point values** for each question are indicated in [**square brackets**].

• As always, it is very important to **EXPLAIN** all of your solutions clearly.

THE QUESTIONS:

1. [**5**] Let $S = \{1, 2, 3, 4\}$, and $\pi_1 = 1/6$, $\pi_2 = 1/12$, and $\pi_3 = 1/2$, $\pi_4 = 1/4$. Find explicit transition probabilities $\{p_{ij}\}_{i,j \in S}$ for a Markov chain on S, with $p_{ij} = 0$ whenever $|i - j| \geq 2$, such that (with proof) $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$ for all $i, j \in S$. [Hint: Don't forget Metropolis.]

2. [**6**] Consider a graph with vertex set $V = \{1, 2, 3, 4, 5\}$, and edge weights $w(1, 2) = w(2, 1) = w(1, 3) = w(3, 1) = w(2, 3) = w(3, 2) = w(1, 4) = w(4, 1) = w(1, 5) = w(5, 1) = w(4, 5) = w(5, 4) = 1$, and $w(u, v) = 0$ otherwise. Let $\{X_n\}$ be random walk on this graph, with $X_0 = 1$. For each of the following limits, determine whether or not the limit exists, and if yes then what it equals: **(a)** $\lim_{n\to\infty} \mathbf{P}(X_n = 1)$, **(b)** $\lim_{n\to\infty} \frac{1}{2}[\mathbf{P}(X_n = 1) + \mathbf{P}(X_{n+1} = 1)]$, and **(c)** $\lim_{n\to\infty} \frac{1}{3}[\mathbf{P}(X_n = 1) + \mathbf{P}(X_{n+1} = 1) + \mathbf{P}(X_{n+2} = 1)]$.

3. [**8**] Suppose each car that passes is independently equally likely to be Red, Green, or Blue. Let τ be the number of cars which pass until the first time we see the pattern Red-Green (i.e., until a Green car passes right after a Red car). Compute $z = \mathbf{E}(\tau)$.

4. **[4]** In the previous question, suppose each Red car is worth 12, each Green car is worth 6, and each Blue car is worth 3. Let Y be the total worth of all cars up to and including time τ. Compute $\mathbf{E}(Y)$. [Note: If you could not solve the previous question, then you may leave your answer to this question in terms of the unknown value "z".]

5. **[6]** Find Markov chain transitions $\{p_{ij}\}_{i,j \in S}$ on the state space $S = \{1, 2, 5\}$ (so there are only 3 states), such that $p_{21} = 1/2$, and also the chain is a martingale.

6. **[6]** Consider a Markov chain $\{X_n\}$ with state space $S = \{5, 6, 7, 8, \ldots\}$, with $p_{5,5} = 1$, and $p_{i,i-1} = p_{i,i} = p_{i,i+1} = 1/3$ for all $i \geq 6$, and with $X_0 = 8$. (You may assume that $\mathbf{E}|X_n| < \infty$ for each n.) Then **(a)** show that as $n \to \infty$, the values $\{X_n\}$ converge with probability 1 to some random variable X, and **(b)** determine whether or not $\mathbf{E}(X) = \lim_{n\to\infty} \mathbf{E}(X_n)$.

7. **[9]** Consider a branching process with initial value $X_0 = 2$, and with offspring distribution given by $\mu\{0\} = 1/3$ and $\mu\{2\} = 2/3$. Let q be the probability of eventual extinction. Then **(a)** compute $\mathbf{P}(X_1 = 2)$, and **(b)** compute $\mathbf{P}(X_2 = 8)$, and **(c)** determine (with explanation) whether $q = 0$, or $q = 1$, or $0 < q < 1$.

8. **[6]** Suppose a stock price X_0 at time 0 is equal to 10, and at time S is random with $\mathbf{P}(X_S = 4) = 3/5$ and $\mathbf{P}(X_S = 12) = 2/5$. Suppose there is also an option to buy one share of the stock at time S for price $K = 7$, and this option has been priced (by some company) at the value \$2. Assume you are allowed to buy or sell any amount of the stock at its given price of 10, and also to buy or sell any amount of this option at its listed price of 2. Determine (with explanation) an explicit amount of stock and of this option that you could buy or sell at time 0 to achieve arbitrage, i.e. to make a guaranteed positive profit.

[END OF EXAMINATION; total points = 50]