STA 2111 (Graduate Probability I), Fall 2026

Suggested Homework #1. Optional; *NOT* to be handed in.

1. Let $\Omega = \mathbf{Z}$ be the set of all integers, and let

$$\mathcal{B} \;=\; \{A \subseteq \Omega : \text{either } A \text{ is finite or } A^C \text{ is finite}\}\,.$$

Let $\mathbf{P} : \mathcal{B} \to [0,1]$ by $\mathbf{P}(A) = 0$ if A is finite, and $\mathbf{P}(A) = 1$ if A^C is finite.

(a) [**5**] Determine whether $\mathcal{B}$ is an algebra (meaning that $\emptyset, \Omega \in \mathcal{B}$, and $\mathcal{B}$ is closed under complement and under finite union).

(b) [**5**] Determine whether $\mathcal{B}$ is a σ-algebra.

(c) [**5**] Determine whether $\mathbf{P}$ is finitely additive on $\mathcal{B}$.

(d) [**5**] Determine whether $\mathbf{P}$ is countably additive on $\mathcal{B}$ (meaning that if $A_1, A_2, \ldots \in \mathcal{B}$ are disjoint, and if also $\bigcup_n A_n \in \mathcal{B}$, then $\mathbf{P}(\bigcup_n A_n) = \sum_n \mathbf{P}(A_n)$).

2. Let $\Omega = \{1,2,3,4\}$, and let $\mathcal{J} = \Big\{\emptyset, \{1\}, \{2\}, \{3,4\}, \Omega\Big\}$. Define $\mathbf{P} : \mathcal{J} \to [0,1]$ by $\mathbf{P}(\emptyset) = 0$, $\mathbf{P}\{1\} = 1/7$, $\mathbf{P}\{2\} = 2/7$, $\mathbf{P}\{3,4\} = 4/7$, and $\mathbf{P}(\Omega) = 1$.

(a) [**3**] Prove that $\mathcal{J}$ is a semi-algebra.

(b) [**4**] Find $\mathbf{P}^*(A)$ and $\mathbf{P}^*(A^C)$, where $A = \{2,3\} \subseteq \Omega$ and $\mathbf{P}^*$ is outer measure.

(c) [**4**] Determine whether or not $A \in \mathcal{M}$, where $\mathcal{M}$ is the σ-algebra constructed in the proof of the Extension Theorem. [Hint: Perhaps consider the case $E = \Omega$.]

3. [**5**] Prove that the extension $(\Omega, \mathcal{M}, \mathbf{P}^*)$ constructed in the proof of the Extension Theorem must be "complete", i.e. if $A \in \mathcal{M}$ with $\mathbf{P}^*(A) = 0$, and $B \subseteq A$, then $B \in \mathcal{M}$. (It follows from monotonicity that $\mathbf{P}^*(B) = 0$.)

4. Let $\Omega = \{1,2,3,4\}$, and $\mathcal{F} = 2^\Omega$ the collection of all subsets of Ω, and

$$\mathcal{C} \;=\; \Big\{\emptyset, \{1,2\}, \{2,3\}, \{3,4\}, \Omega\Big\}\,.$$

Define functions $\mu, \nu : \mathcal{F} \to [0,1]$ by $\mu(A) = \frac{1}{2}\mathbf{1}_A(1) + \frac{1}{2}\mathbf{1}_A(3)$ and $\nu(A) = \frac{1}{2}\mathbf{1}_A(2) + \frac{1}{2}\mathbf{1}_A(4)$, where e.g. $\mathbf{1}_A(3) = 1$ if $3 \in A$ or $\mathbf{1}_A(3) = 0$ if $3 \notin A$.

(a) [**5**] Prove that $(\Omega, \mathcal{F}, \mu)$ is a valid probability triple. (It then follows similarly that $(\Omega, \mathcal{F}, \nu)$ is also a valid probability triple.)

(b) [**3**] Determine whether $\mathcal{C}$ is an algebra.

(c) [**3**] Determine whether $\mathcal{C}$ is a semi-algebra.

(d) [**5**] Find $\sigma(\mathcal{C})$, the smallest σ-algebra containing all elements of $\mathcal{C}$.

(e) [**3**] Determine whether $\mu(A) = \nu(A)$ for all $A \in \mathcal{C}$.

(f) [**3**] Determine whether $\mu(A) = \nu(A)$ for all $A \in \mathcal{F}$.

(g) **[3]** Explain why these facts do not contradict our theorem about uniqueness of extensions of probability measures.

5. **[4]** Suppose that $\Omega = \{1, 2\}$, and $\mathcal{F} = 2^{\Omega}$ is the collection of all subsets of Ω, and $\mathbf{P} : \mathcal{F} \to [0, 1]$ with $\mathbf{P}(\emptyset) = 0$ and $\mathbf{P}(\Omega) = 1$. Prove that $\mathbf{P}$ is countably additive if and only if $\mathbf{P}\{1\} \geq 0$, $\mathbf{P}\{2\} \geq 0$, and $\mathbf{P}\{1\} + \mathbf{P}\{2\} = 1$.

6. Let $\Omega = \{1, 2, 3, 4\}$. Determine whether or not each of the following is a σ-algebra.

(a) **[4]** $\mathcal{F}_1 = \Big\{\emptyset,\ \{1,3\},\ \{2,4\},\ \{1,2,3,4\}\Big\}$.

(b) **[4]** $\mathcal{F}_2 = \Big\{\emptyset,\ \{1,2\},\ \{1,3\},\ \{1,4\},\ \{2,3\},\ \{2,4\},\ \{3,4\},\ \{1,2,3,4\}\Big\}$.

(c) **[3]** $\mathcal{F}_3 = \Big\{\emptyset,\ \{3\},\ \{4\},\ \{1,2\},\ \{3,4\},\ \{1,2,3\},\ \{1,2,4\},\ \{1,2,3,4\}\Big\}$.

7. **[5]** Suppose that $\Omega = \mathbf{Z}$ is the set of all integers, and $\mathbf{P}$ is defined for all $A \subseteq \Omega$ by $\mathbf{P}(A) = 0$ if A is finite, and $\mathbf{P}(A) = 1$ if A is infinite. Is $\mathbf{P}$ finitely additive?

8. **[5]** Let $\mathcal{F}$ be a collection of subsets of Ω, such that $\Omega \in \mathcal{F}$. Suppose that whenever $A, B \in \mathcal{F}$, then also $A \setminus B \equiv A \cap B^C \in \mathcal{F}$. Prove that $\mathcal{F}$ is an algebra.

9. For any interval $I \subseteq [0, 1]$, let $\mathbf{P}(I)$ be the length of I.

(a) **[5]** Prove that if $I_1, I_2, \ldots, I_n$ is a finite collection of intervals, and if $\bigcup_{j=1}^{n} I_j \supseteq I_*$ for some interval I_*, then $\sum_{j=1}^{n} \mathbf{P}(I_j) \geq \mathbf{P}(I_*)$. [Hint: Suppose I_j has left endpoint a_j and right endpoint b_j, and first re-order the intervals so $a_1 \leq a_2 \leq \ldots \leq a_n$.]

(b) **[5]** Prove that if $I_1, I_2, \ldots$ is a countable collection of open intervals, and if $\bigcup_{j=1}^{\infty} I_j \supseteq I_*$ for some closed interval I_*, then $\sum_{j=1}^{\infty} \mathbf{P}(I_j) \geq \mathbf{P}(I_*)$. [Hint: You may use the Heine-Borel Theorem, which says that if a collection of open intervals contain a closed interval, then some finite sub-collection of the open intervals also contains the closed interval.]

(c) **[5]** Prove that if $I_1, I_2, \ldots$ is *any* countable collection of intervals, and if $\bigcup_{j=1}^{\infty} I_j \supseteq I_*$ for *any* interval I_*, then $\sum_{j=1}^{\infty} \mathbf{P}(I_j) \geq \mathbf{P}(I_*)$. [Hint: Extend the interval I_j by $\epsilon\, 2^{-j}$ at each end, and decrease I_* by ϵ at each end, while making I_j open and I_* closed. Then use part (b).] (Note: This is the "countable monotonicity" property needed to apply the Extension Theorem for the Uniform[0,1] distribution, to guarantee that $\mathbf{P}^*(I) \geq \mathbf{P}(I)$.)

(d) **[4]** Suppose we instead defined $\mathbf{P}(I)$ to be the square of the length of I. Show that in that case, the conclusion of part (c) would not hold.