

STA 2111 (Graduate Probability I), Fall 2026

Suggested Homework #2. Optional; *NOT* to be handed in.

1. [5] Let $A_1, A_2, \dots$ be independent events. Let Y be a random variable which is measurable with respect to $\sigma(A_n, A_{n+1}, \dots)$ for each $n \in \mathbf{N}$. Prove there is some $x \in \mathbf{R}$ such that $\mathbf{P}(Y = x) = 1$. [Hints: You may wish to consider $\mathbf{P}(Y \leq y)$. And don't forget Kolmogorov.]
2. Let X be a non-negative random variable with $\mathbf{P}(X > 0) > 0$.
 - (a) [4] Prove that there must exist some $n \in \mathbf{N}$ such that $\mathbf{P}(X \geq 1/n) > 0$.
 - (b) [4] Prove that we must have $\mathbf{E}(X) > 0$.
3. For each of the following conditions, either give an example of a random variable X defined on Lebesgue measure on $[0, 1]$ satisfying those conditions, or prove that no such random variable exists.
 - (a) [3] $X \geq 0$, and $\mathbf{E}(X) = 3$, and $\mathbf{P}(X \geq 7) = 1/2$.
 - (b) [3] $\mathbf{E}(X) = 3$, and $\mathbf{P}(X \geq 7) = 1/2$.
 - (c) [3] $\mathbf{E}(X) = 3$, and $\mathbf{Var}(X) = 4$, and $\mathbf{P}(X \leq 0) = 1/2$.
4. Let Y be any non-negative random variable.
 - (a) [3] Prove that $Y \mathbf{1}_{Y>0} = Y$. [Hint: Consider separately $Y(\omega) = 0$ and $Y(\omega) > 0$.]
 - (b) [5] Assuming that $0 < \mathbf{E}(Y^2) < \infty$, prove that $\mathbf{P}(Y > 0) \geq [\mathbf{E}(Y)]^2 / \mathbf{E}(Y^2)$. [Hints: apply Cauchy-Schwarz with $X = \mathbf{1}_{Y>0}$. And don't forget part (a).]
5. Suppose X and $\{X_n\}$ are all random variables defined on some $(\Omega, \mathcal{F}, \mathbf{P})$ where Ω is countable and $\mathcal{F} = 2^\Omega$, and that $\{X_n\} \rightarrow X$ in probability.
 - (a) [4] Prove that if $\omega \in \Omega$ with $\mathbf{P}(\{\omega\}) > 0$, then for all $\epsilon > 0$, there is $N \in \mathbf{N}$ such that $\mathbf{P}(|X_n - X| \geq \epsilon) < \mathbf{P}(\{\omega\})$ for all $n \geq N$.
 - (b) [4] Prove that $\{X_n\} \rightarrow X$ almost surely. [Hint: Part (a) might help.]
6. Let $\{I_j\}_{j=1}^\infty$ be independent random variables, with $I_1 \sim \text{Uniform}\{1, 2, \dots, 9\}$, $I_2 \sim \text{Uniform}\{10, 11, 12, \dots, 99\}$, $I_3 \sim \text{Uniform}\{100, 101, 102, \dots, 999\}$, and in general, I_j is chosen uniformly from the set of all positive integers having j digits. Then, define $\{X_n\}_{n=1}^\infty$ by: $X_n = 7$ if $n = I_j$ for some j , otherwise $X_n = 5$.
 - (a) [4] Prove or disprove that $\{X_n\} \rightarrow 5$ in probability.
 - (b) [4] Prove or disprove that $\{X_n\} \rightarrow 5$ almost surely.
7. Let $\{X_n\}$ be random variables defined on $\text{Uniform}[0, 1]$, with $|X_n(\omega)| < \infty$ and $\lim_{n \rightarrow \infty} X_n(\omega) = X(\omega)$ for each fixed $\omega \in [0, 1]$. Suppose for each $n \in \mathbf{N}$, $\mathbf{E}|X_n| < \infty$, and also $X_{n+1}(\omega) \leq X_n(\omega)$ for all $\omega \leq 1/2$, and $X_{n+1}(\omega) \geq X_n(\omega)$ for all $\omega > 1/2$.
 - (a) [3] Prove that $\lim_{n \rightarrow \infty} \mathbf{E}(X_n \mathbf{1}_{(\frac{1}{2}, 1]}) = \mathbf{E}(X \mathbf{1}_{(\frac{1}{2}, 1]})$.

(b) [5] Prove that $\lim_{n \rightarrow \infty} \mathbf{E}(X_n \mathbf{1}_{[0, \frac{1}{2}]}) = \mathbf{E}(X \mathbf{1}_{[0, \frac{1}{2}]})$. [Hint: What can you say about $Y_n = X_1 \mathbf{1}_{[0, \frac{1}{2}]} - X_n \mathbf{1}_{[0, \frac{1}{2}]}$? And, remember that $\mathbf{E}|X_1| < \infty$.]

(c) [3] Assuming $\mathbf{E}|X| < \infty$, prove that $\lim_{n \rightarrow \infty} \mathbf{E}(X_n) = \mathbf{E}(X)$.

(d) [5] Without assuming that $\mathbf{E}|X| < \infty$, find an example of such $\{X_n\}$ where $\mathbf{E}(X)$ is undefined, i.e. $\mathbf{E}(X^+) = \mathbf{E}(X^-) = \infty$, and hence $\lim_{n \rightarrow \infty} \mathbf{E}(X_n) \neq \mathbf{E}(X)$.

8. Let $\Omega = \mathbf{N} \times \{-1, 1\} = \{(i, j) : i \in \mathbf{N}, j = -1 \text{ or } 1\}$, with $\mathcal{F} = 2^\Omega = \underline{\text{all}}$ subsets of Ω , and $\mathbf{P}(i, j) = 2^{-i-1}$ for all $(i, j) \in \Omega$. Let $X_n(i, j) = (j 3^i + 1) \mathbf{1}_{i=n}$ i.e. $X_n(n, 1) = 3^n + 1$ and $X_n(n, -1) = -3^n + 1$, with $X_n(i, j) = 0$ whenever $i \neq n$. Prove or disprove that:

(a) [4] $X_n \rightarrow 0$ in probability.

(b) [4] $X_n \rightarrow 0$ almost surely.

(c) [4] The $\{X_n\}$ are monotonically increasing, i.e. $X_n \leq X_{n+1}$ for all $n \in \mathbf{N}$.

(d) [4] The $\{X_n\}$ are bounded, i.e. there is $M < \infty$ such that $|X_n| \leq M$ for all $n \in \mathbf{N}$.

(e) [4] The $\{X_n\}$ are dominated, i.e. there is a random variable Y on $(\Omega, \mathcal{F}, \mathbf{P})$ with $\mathbf{E}|Y| < \infty$ such that $|X_n| \leq Y$ for all $n \in \mathbf{N}$.

(f) [4] The $\{X_n\}$ are uniformly integrable, i.e. $\lim_{\alpha \rightarrow \infty} \sup_n \mathbf{E}(|X_n| \mathbf{1}_{|X_n| \geq \alpha}) = 0$.

(g) [4] $\lim_{n \rightarrow \infty} \mathbf{E}(X_n) = 0$.

9. Let $\Omega = \mathbf{N}$, $\mathcal{F} = 2^\Omega$, and $\mathbf{P}(\omega) = 2^{-\omega}$. Let $Y_n(\omega) = 2^n \mathbf{1}_n(\omega)$. Also, let $W = \sum_{n=1}^{\infty} (-1)^{n-1} Y_n \equiv Y_1 - Y_2 + Y_3 - Y_4 + Y_5 - Y_6 + \dots$

(a) [3] Compute the expected value $\mathbf{E}(Y_n)$ for each n .

(b) [4] Compute the expected values $\mathbf{E}(W^+)$ and $\mathbf{E}(W^-)$.

(c) [3] What can be said about the expected value $\mathbf{E}(W)$?

[END; total points = 100]