STA357 Background Facts, Fall 2026

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This document provides a few background mathematics and probability facts which will be used in STA357. (Note that students are responsible for knowing all needed background, at the level of STA257 or Chapters 1-4 of the Evans-Rosenthal book.)

This document might be updated throughout the semester.

Continuity of Probabilities

Write $\{A_n\} \nearrow A$ to mean $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ and $\bigcup_n A_n = A$. ("A_n increases to A")

Write $\{A_n\} \searrow A$ to mean $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ and $\bigcap_n A_n = A$. ("A_n decreases to A")

Note that $\{A_n\} \searrow A$ iff $\{A_n^C\} \nearrow A^C$.

Fact: If $\{A_n\} \nearrow A$ or $\{A_n\} \searrow A$, then $\lim_{n\to\infty} \mathbf{P}(A_n) = \mathbf{P}(A)$.

Proof: Suppose $\{A_n\} \nearrow A$. Let $B_n = A_n \cap A_{n-1}^C$. Then the $\{B_n\}$ are disjoint, with $\bigcup B_n = \bigcup A_n = A$. Hence,

$$\mathbf{P}(A) = \mathbf{P}\Big(\bigcup_{m=1}^{\infty} B_m\Big) = \sum_{m=1}^{\infty} \mathbf{P}(B_m) = \lim_{n\to\infty} \sum_{m=1}^{n} \mathbf{P}(B_m) = \lim_{n\to\infty} \mathbf{P}\Big(\bigcup_{m\le n} B_m\Big) = \lim_{n\to\infty} \mathbf{P}(A_n)\,.$$

If instead $\{A_n\} \searrow A$, then $\{A_n^C\} \nearrow A^C$, so

$$\mathbf{P}(A) = 1 - \mathbf{P}(A^C) = 1 - \lim_{n\to\infty} \mathbf{P}(A_n^C) = \lim_{n\to\infty} \big(1 - \mathbf{P}(A_n^C)\big) = \lim_{n\to\infty} \mathbf{P}(A_n)\,. \qquad \blacksquare$$

Liminf and Limsup

For a sequence $x_1, x_2, \ldots$ of real numbers, we define:

$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf_{k\ge n} x_k$ and $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{k\ge n} x_k$.

These limits always exist (though they may be $\pm\infty$).

Furthermore, $\lim_{n\to\infty} x_n$ exists if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$, in which case $\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

de Morgan's Laws and Set Theory

We always have $\left(\bigcup_\alpha A_\alpha\right)^C = \bigcap_\alpha A_\alpha^C$ and $\left(\bigcap_\alpha A_\alpha\right)^C = \bigcup_\alpha A_\alpha^C$.

That is, complement converts unions to intersections, and vice-versa.

Also, we can interpret $\bigcap_\alpha$ as "for all α", and $\bigcup_\alpha$ as "there exists α".

For example, "For all $n \in \mathbf{N}$, there is $k \geq n$ such that A_n occurs" is equivalent to:

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_n\,.$$

And, "For all $\epsilon > 0$, there is $N \in \mathbf{N}$ such that $x_n < \epsilon$ for all $n \geq N$" is equivalent to:

$$\bigcap_{\epsilon>0} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{x_n < \epsilon\}\,.$$

Exponential Bound

Fact: For any real number x (even negative), we always have $e^x \geq 1 + x$.

Proof: Let $f(x) = e^x - 1 - x$.

Then $f(0) = 0$, and $f'(0) = 0$. And, $f''(x) = e^x > 0$ for all x.

For $u > 0$, $f'(u) = f'(u) - f'(0) = \int_0^u f''(v)\,dv > 0$. Hence, for $x > 0$, $f(x) = \int_0^x f'(u)\,du > 0$.

For $u < 0$, $-f'(u) = f'(0) - f'(u) = \int_u^0 f''(v)\,dv > 0$, so $f'(u) < 0$. Hence, for $x < 0$, $-f(x) = f(0) - f(x) = \int_x^0 f'(u)\,du < 0$, so $f(x) > 0$.

Hence, in all cases, $f(x) \geq 0$, i.e. $e^x \geq 1 + x$. ∎

Rational Replacement

Consider a union such as $\left\{\bigcup_{\epsilon>0} |Z_n - Z| \geq \epsilon \ i.o.\right\}$.

Is it equivalent to only take the union over rational $\epsilon > 0$? Yes!

Why? Well, clearly $\left\{\bigcup_{\epsilon>0} |Z_n - Z| \geq \epsilon \ i.o.\right\} \supseteq \mathbf{P}\Big(\bigcup_{\substack{\epsilon'>0 \\ \epsilon' \text{rational}}} |Z_n - Z| \geq \epsilon \ i.o.\Big)$.

But given <u>any</u> $\epsilon > 0$, there exists a <u>rational</u> $\epsilon' \in (0, \epsilon)$. For this ϵ', we have that

$$\{|Z_n - Z| \geq \epsilon \ i.o.\} \ \subseteq \ \{|Z_n - Z| \geq \epsilon' \ i.o.\} \ .$$

It follows that $\bigcup_{\epsilon > 0} \Big\{|Z_n - Z| \geq \epsilon \ i.o.\Big\} \ \subseteq \ \bigcup_{\substack{\epsilon' > 0 \\ \epsilon' \text{rational}}} \Big\{|Z_n - Z| \geq \epsilon' \ i.o.\Big\}$.

Hence, we actually have $\bigcup_{\epsilon > 0} \Big\{|Z_n - Z| \geq \epsilon \ i.o.\Big\} \ = \ \bigcup_{\substack{\epsilon' > 0 \\ \epsilon' \text{rational}}} \Big\{|Z_n - Z| \geq \epsilon' \ i.o.\Big\}$. ∎