STA357 Suggested Exercises, Fall 2026

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This document provides some suggested homework exercises for some of the topics covered in STA357, to help prepare for the tests. They are not to be handed in.

This document might be updated throughout the semester.

These exercises are taken from various other books, modified as necessary.

Note that some of these exercises might be harder than typical test questions, so please do not feel discouraged if you find some of them challenging.

Infinitely Often (i.o.) and Borel-Cantelli

Exercise: Let $\mathbf{P}$ be the uniform distribution on $\{1, 2, 3\}$. Give an example of a sequence of events $A_1, A_2, \ldots$ such that $\limsup_{n\to\infty} \mathbf{P}(A_n) < \mathbf{P}(A_n \; i.o.)$, i.e. the inequality is strict.

Exercise: Let $\mathbf{P}$ be the Uniform distribution on $[0, 1]$, and let $0 \le c \le d \le 1$ be arbitrary real numbers. Give an example of a sequence $A_1, A_2, \ldots$ of subsets of $[0, 1]$, such that $\limsup_{n\to\infty} \mathbf{P}(A_n) = c$, and $\mathbf{P}(A_n \; i.o.) = d$.

Exercise: Let $A_1, A_2, \ldots, B_1, B_2, \ldots$ be events, and let $C_n = A_n \cap B_n$.
(a) Prove that $\{A_n \; i.o.\} \cap \{B_n \; i.o.\} \;\supseteq\; \{C_n \; i.o.\}$.
(b) Give an example where this inclusion is strict, and another example where it is equal.

Exercise: Let $A_1, A_2, \ldots$ be a sequence of events, and let $N \in \mathbf{N}$. Suppose there are events B and C such that $B \subseteq A_n \subseteq C$ for all $n \ge N$, and such that $\mathbf{P}(B) = \mathbf{P}(C)$. Prove that $\mathbf{P}(A_n \; i.o.) = \mathbf{P}(B) = \mathbf{P}(C)$.

Exercise: Let $\{X_n\}_{n=1}^{\infty}$ be independent random variables, with $\mathbf{P}(X_n = i) = 1/n$ for $i = 1, 2, \ldots, n$. Compute $\mathbf{P}(X_n = 5 \; i.o.)$, the probability that an infinite number of the X_n are equal to 5.

Exercise: Let $X_1, X_2, \ldots$ be jointly-defined random variables, with $\sum_{i=1}^{\infty} i^2 \; \mathbf{P}(i \le X_n < i + 1) \le C < \infty$ for all n. Prove that $\mathbf{P}(X_n \ge n \; i.o.) = 0$.

Exercise: Let X be a random variable with $\mathbf{P}(X > 0) > 0$. Prove that there is $\delta > 0$ such that $\mathbf{P}(X \ge \delta) > 0$. [Hint: Don't forget continuity of probabilities.]

Exercise: Let $\delta, \epsilon > 0$, and let $X_1, X_2, \ldots$ be a sequence of independent non-negative random variables such that $\mathbf{P}(X_i \ge \delta) \ge \epsilon$ for all i. Prove that $\mathbf{P}\Big(\sum_{i=1}^{\infty} X_i = \infty\Big) = 1$.

Exercise: Let $A_1, A_2, \ldots$ be a sequence of events, such that (i) $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ are independent whenever $i_{j+1} \geq i_j + 2$ for $1 \leq j \leq k-1$, and (ii) $\sum_n \mathbf{P}(A_n) = \infty$. Then the Borel-Cantelli Lemma does not directly apply. Still, prove that $\mathbf{P}(A_n \ i.o.) = 1$.

Exercise: Consider infinite, independent, fair coin tossing, and let H_n be the event that the n^{th} coin is heads. Determine the following probabilities.
(a) $\mathbf{P}(H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{n+9} \ i.o.)$.
(b) $\mathbf{P}(H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{2n} \ i.o.)$.
(c) $\mathbf{P}(H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{n+\lfloor 2 \log_2 n \rfloor} \ i.o.)$.
(d) $\mathbf{P}(H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{n+\lfloor \log_2 n \rfloor} \ i.o.)$. [Hint: Consider $\{A_{2n\lfloor \log_2 n \rfloor}\}$.]

Convergence in Probability and WLLN

Exercise: Let $U \sim$ Uniform$[5, 10]$, and $Z = \mathbf{1}_{U \in [5,7)}$, and $Z_n = \mathbf{1}_{U \in [5, 7+\frac{1}{n^2})}$. Prove that $Z_n \xrightarrow{P} Z$.

Exercise: Let $Y \sim$ Uniform$[0, 1]$, and $X_n = Y^n$. Prove that $X_n \xrightarrow{P} 0$.

Exercise: Let $W_1, W_2, \ldots$ be i.i.d. $\sim$ Exponential(3). Prove that there is some $n \in \mathbf{N}$ with $\mathbf{P}(W_1 + W_2 + \ldots + W_n < n/2) > 0.999$.

Exercise: Let $Y_1, Y_2, \ldots$ be i.i.d. $\sim$ Normal(2, 5). Prove that there is some $n \in \mathbf{N}$ with $\mathbf{P}(Y_1 + Y_2 + \ldots + Y_n > n) > 0.999$.

Exercise: Let $X_1, X_2, \ldots$ be i.i.d. $\sim$ Poisson(8). Prove that there is some $n \in \mathbf{N}$ with $\mathbf{P}(X_1 + X_2 + \ldots + X_n > 9n) > 0.999$.

Exercise: Suppose $U \sim$ Uniform$[0, 1]$ and $Y_n = \frac{n-1}{n} U$. Prove that $Y_n \xrightarrow{P} U$.

Exercise: Let C_n be the number of heads when flipping n coins, and $X_n = e^{-C_n}$. Prove that $X_n \xrightarrow{P} 0$.

Exercise: Let C_n be the number of heads when flipping n coins, and $Y_n = C_n/(C_n + 1)$. Prove that $Y_n \xrightarrow{P} 1$.

Exercise: Let $Z_n \sim$ Uniform$[0, n]$, and $W_n = 5Z_n/(Z_n + 1)$. Prove that $W_n \xrightarrow{P} 5$.

Exercise: Let Z_n be the sum of the squares of the numbers showing when rolling n fair six-sided dice. Find some number m such that $\frac{1}{n} Z_n \xrightarrow{P} m$.

Exercise: Consider flipping n nickels and n dimes. Let X_n equal 4 times the number of nickels showing heads, plus 5 times the number of dimes showing heads. Find some number r such that $\frac{1}{n} X_n \xrightarrow{P} r$.

Exercise: Provide an example of random variables X_n such that $X_n \xrightarrow{P} 0$, but $\mathbf{E}(X_n) = 1$ for all n.

Exercise: Prove or disprove that $X_n \xrightarrow{P} 0$ if and only if $|X_n| \xrightarrow{P} 0$.

Exercise: Prove or disprove that $X_n \xrightarrow{P} 5$ if and only if $|X_n| \xrightarrow{P} 5$.

Exercise: Suppose $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$. Prove that $X_n + Y_n \xrightarrow{P} X + Y$.

Exercise: Let $X_1, X_2, \ldots$ be a sequence of random variables, with $\mathbf{E}[X_n] = 8$ and $\mathrm{Var}[X_n] = 1/\sqrt{n}$ for each n. Prove or disprove that $\{X_n\}$ must converge to 8 in probability.

Exercise: Let $r \in \mathbf{N}$. Let $X_1, X_2, \ldots$ be identically distributed random variables having finite mean m, which are r-*dependent*, i.e. such that $X_{k_1}, X_{k_2}, \ldots, X_{k_j}$ are independent whenever $k_{i+1} > k_i + r$ for each i. (Thus, independent random variables are 0-dependent.) Prove that $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} m$ as $n \to \infty$. [Hint: Break up the sum $\sum_{i=1}^{n} X_i$ into $r+1$ different sums.]

Exercise: Let $X_1, X_2, \ldots$ be a sequence of independent random variables with $\mathbf{P}(X_n = 3^n) = \mathbf{P}(X_n = -3^n) = \frac{1}{2}$. Let $S_n = X_1 + \ldots + X_n$.
(a) Compute $\mathbf{E}(X_n)$ for each n.
(b) For $n \in \mathbf{N}$, compute $R_n \equiv \sup\{r \in \mathbf{R};\ \mathbf{P}(|S_n| \geq r) = 1\}$, i.e. the largest number such that $|S_n|$ is always at least R_n.
(c) Compute $\lim_{n\to\infty} \frac{1}{n} R_n$.
(d) For which $\epsilon > 0$ (if any) is it the case that $\mathbf{P}(\frac{1}{n}|S_n| \geq \epsilon) \not\to 0$?
(e) Why does this result not contradict the weak law of large numbers?