1953: The Metropolis Algorithm. A *Markov chain* is a random sequence of states, each of whose probabilities depend iteratively on the previous state. Metropolis et al. realised in 1953 that Markov chains could be run on then-new electronic computers to converge to, and hence sample from, a probability distribution of interest. Consider the special case where the set of possible states is equal to the integers, $\mathbf{Z}$. Let $\{\pi_i\}_{i \in \mathbf{Z}}$ be any positive probability distribution on S, i.e. a collection of real numbers $\pi_i > 0$ with $\sum_{i \in \mathbf{Z}} \pi_i = 1$. Let $\{p_{i,j}\}_{i,j \in \mathbf{Z}}$ be Markov chain transition probabilities, so $p_{i,j}$ equals the probability, given that the state at time n equals i, that the state at time $n+1$ will equal j. The question is, can we find simple transition probabilities $p_{i,j}$, such that the chain "converges to π", i.e. for each $i \in \mathbf{Z}$, the probability that the state at time n is equal to i converges, as $n \to \infty$, to π_i.

In fact, the answer is yes! For $i \in \mathbf{Z}$, let $p_{i,i+1} = \frac{1}{2} \min[1, \frac{\pi_{i+1}}{\pi_i}]$, $p_{i,i-1} = \frac{1}{2} \min[1, \frac{\pi_{i-1}}{\pi_i}]$, and $p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1}$, with $p_{i,j} = 0$ otherwise. Then this Markov chain is easily run on a computer (for an animated version see e.g. www.probability.ca/met), and has good convergence properties as the following problem shows. It and various generalizations have led to the explosive growth of *Markov chain Monte Carlo (MCMC) algorithms*, which have revolutionized subjects from statistical physics to Bayesian inference to theoretical computer science to financial mathematics.

Problem: Show that the above Markov chain:
(a) is *irreducible*, i.e. for any $i, j \in \mathbf{Z}$ there are $m \in \mathbf{N}$ and $k_1, \ldots, k_m \in \mathbf{Z}$ such that $p_{i,k_1} > 0$ and $p_{k_m,j} > 0$ and $p_{k_n,k_{n+1}} > 0$ for $1 \le n \le m-1$.
(b) is *aperiodic*, in particular there is at least one $i \in \mathbf{Z}$ with $p_{i,i} > 0$.
(c) is *reversible*, i.e. $\pi_i p_{i,j} = \pi_j p_{j,i}$ for all $i, j \in \mathbf{Z}$.
(d) leaves π *stationary*, i.e. $\sum_{i \in \mathbf{Z}} \pi_i p_{i,j} = \pi_j$ for all $j \in \mathbf{Z}$. [Hint: Use part (c).]
(e) *converges to* π as described above. [Hint: This follows from parts (a), (b), and (d) by the standard Markov chain convergence theorem, see e.g. Section 1.8 of Norris (1998).]

References: N. Metropolis, A. Rosenbluth, M. Rosenbluth, A. Teller, and E. Teller, Equations of state calculations by fast computing machines. J. Chem. Phys. **21** (1953), 1087–1091.

J.R. Norris, Markov Chains. Cambridge University Press, 1998. Available at: http://www.statslab.cam.ac.uk/~james/Markov/