Is it better to be born Rich or Lucky?

The mathematics of random walks.

The set-up. Let a and ℓ be integers, with $1 \le a \le 7$ and $2 \le \ell \le 5$.

The game. You start with $\$a$. You then repeatedly roll a fair 6-sided die. Each time the die shows at least ℓ, you win \$1. Each time the die shows less than ℓ, you lose \$1. The game continues until such time as your fortune either reaches \$8 (in which case you WIN), or reaches \$0 (in which case you LOSE).

The question. Which of the following gives you the best chance of winning?

(a) ("Rich") Here $a = 6$ and $\ell = 5$, so you start with \$6, but have only a 1/3 chance of winning each time.

(b) ("Medium") Here $a = 4$ and $\ell = 4$, so you start with \$4, and have a 1/2 chance of winning each time.

(c) ("Lucky") Here $a = 2$ and $\ell = 3$, so you start with only \$2, but have a 2/3 chance of winning each time.

The experiment. Try it out! Make a board with 9 squares, numbered from 0 to 8. Use a marker to keep track of your current fortune, and repeatedly roll an ordinary die, moving the marker one square each time, to see what happens. Then start over again! Keep a table of the results. How often do each of Rich, Medium, and Lucky seem to be winning?

The mathematics. Now try to compute the probability of winning, as a function of a and p (where $p = (7 - \ell)/6$ is the chance of winning each time). [**Hint:** Let $s_p(a)$ be this chance of winning. Then $s_p(0)$ and $s_p(8)$ are easy. Now, for $1 \le a \le 7$, by considering the two possibilities for the <u>first</u> roll, come up with a "difference equation" giving $s_p(a)$ in terms of $s_p(a+1)$ and $s_p(a-1)$ (and also involving p and $1-p$). Then solve the equation!]

Various other questions. What is the average time the game will take? How do the answers to these questions change if "8" is replaced by "N"? or by "∞"? What is $s_p(a)$ if $N = 1000$, $a = 900$, and $p = 18/38$ like in roulette? Given that you eventually lose, what is the conditional probability that your fortune had gotten as high as \$7?

Jeffrey S. Rosenthal
Department of Statistics
University of Toronto

Teacher's supplement to

Is it better to be born Rich or Lucky?

The solution. Clearly $s_p(0) = 0$ and $s_p(8) = 1$. To set up the difference equation for $1 \le a \le 7$, we reason as follows. On the first roll we could either win (with probability p, so that we now have \$$(a+1)$) or lose (with probability $1-p$, so that we now have \$$(a-1)$). Thus,

$$s_p(a) = p\, s_p(a+1) + (1-p)s_p(a-1)\,,$$

or

$$s_p(a+1) - \frac{1}{p}\, s_p(a) + \frac{1-p}{p}\, s_p(a-1) \;=\; 0\,.$$

This is a linear second-order difference equation. If $p \neq 1/2$, the general solution is $s_p(a) = A + B\left(\frac{1-p}{p}\right)^a$. Since $s_p(0) = 0$ and $s_p(8) = 1$, we must have $A = -B = \frac{1}{1-\left(\frac{1-p}{p}\right)^8}$, whence

$$s_p(a) \;=\; \frac{1-\left(\frac{1-p}{p}\right)^a}{1-\left(\frac{1-p}{p}\right)^8}\,, \qquad p \neq 1/2\,.$$

For Rich, this works out to about 0.2471. For Lucky, this works out to about 0.7529. Thus, it's better to be born Lucky than Rich!

On the other hand, if $p = \frac{1}{2}$, the general solution is $s_{\frac{1}{2}}(a) = A + Ba$. Since $s_{\frac{1}{2}}(0) = 0$ and $s_{\frac{1}{2}}(8) = 1$, we must have $A = 0$ and $B = 1/8$, whence

$$s_{\frac{1}{2}}(a) \;=\; a/8\,.$$

For Medium, this of course works out to exactly 0.5.

Extensions. The exact same method and formulas work if "8" is replaced by "N". If $N = 1000$, $a = 900$, and $p = 18/38$, then $s_p(a)$ is about 2.656×10^{-5}. Thus, in ordinary roulette, you don't have much chance of getting rich!

To replace N by infinity, we can simply take the limit of these formulas as $N \to \infty$. (Formally, this is valid because of a result called "continuity of probabilities".) We see that, if $p \le \frac{1}{2}$, then the probability of eventually going broke is 1. Gamblers beware! (The other questions can be solved by similar methods, but require a bit of thought.)

Discussion. This "gambler's ruin" problem is an example of a "random walk". Algorithms involving randomness are very commonly used to approximately compute things which are too complicated to compute directly. Understanding the long-term behaviour of random processes is very important, and also leads to interesting mathematical questions. As with many topics in probability theory, sophisticated mathematics can be applied to interesting practical problems.