PMU 199Y, L0411, 2010–11: Math assignment about "Gambler's Ruin"

Recall that we are studying the following game. **A** starts with six pennies; **B** starts with two pennies. We repeatedly roll a fair six-sided die. Each time it shows 1 or 2, **B** gives one penny to **A**; each time it shows 3, 4, 5, or 6, **A** gives one penny to **B**. The game continues until one person wins all eight pennies; that person is the "winner".

In this assignment, we shall try to figure out, mathematically, the probability that **A** wins the game. Note that this is <u>not</u> an easy problem, and it will take us a number of weeks and many steps to solve. Working cooperatively with your assigned group, consider the following questions. Make sure that <u>all</u> group members understand the answer to each question. If you get stuck, you may ask the instructor for assistance.

1. What is the probability that **A** wins the game on the <u>second</u> roll of the die?

2. What is the probability that **A** wins the game on the <u>third</u> role of the die? On the <u>fifth</u> roll? On any <u>odd</u> numbered roll?

3. What is the probability that **A** wins the game on the <u>fourth</u> roll of the die? [This is a bit harder, you have to consider several different cases.]

4. What is the probability that **A** wins the game on the <u>sixth</u> roll of the die? [This is still harder, there are more cases to consider.]

5. Do you think it would be possible to continue in this way, for the <u>eighth</u> roll, the <u>tenth</u> roll, etc., and eventually find the <u>total</u> probability that **A** wins the game?

To make progress solving this problem, we will think about the problem in a new way. As a first step, we will let $s(a)$ stand for the probability that **A** wins this game if they start with a pennies (and **B** starts with $8 - a$ pennies). Thus, $s(6)$ is the probability that we actually want to find, while e.g. $s(3)$ is the probability that **A** wins this game if instead they start with 3 pennies and **B** starts with 5 pennies.

6. Make sure everybody in your group understands this notation. For example, what does $s(5)$ stand for? What about $s(4)$?

7. Since all we really want to know is $s(6)$, why do you think we need to introduce all these other numbers $s(a)$ for different values of a?

8. Why is it necessary to use mathematical notation for these probabilities? For example, why is it better to write $s(6)$ than to write out "the probability that **A** wins this game if they start with 6 pennies" each time?

9. Although the probabilities $s(a)$ are difficult to compute for most values of a, there are two probabilities which are very easy to compute, namely $s(0)$ and $s(8)$. What does $s(0)$ equal? What does $s(8)$ equal?

10. Make your *best guess* as to the values of $s(a)$ for all $a = 0, 1, 2, 3, 4, 5, 6, 7, 8$. [**Note:** you can't actually solve for the $s(a)$ values yet; just guess values for them.]

We next relate these different unknown probabilities $s(a)$ to each other.

11. Find a formula for $s(6)$ in terms of $s(5)$ and $s(7)$. That is, suppose we knew both $s(5)$ and $s(7)$. Could we then easily compute $s(6)$? By what formula? [**Hint:** Suppose **A** starts with six pennies. Consider the two possibilities for what happens after the first roll of the die. How many pennies will **A** have then?] **Note:** This question is the key to the whole exercise. It will require careful thought, though the resulting formula is not complicated.

12. Similarly, find a formula for $s(3)$ in terms of $s(2)$ and $s(4)$. Also find a formula for $s(1)$ in terms of $s(0)$ and $s(2)$.

13. More generally, find a formula for $s(a)$ in terms of $s(a-1)$ and $s(a+1)$, for any integer a between 1 and 7 inclusive.

At this point, we have seven unknowns, namely $s(1)$, $s(2)$, $s(3)$, $s(4)$, $s(5)$, $s(6)$, and $s(7)$. We also have seven equations, namely the seven equations from the previous question (one for each different value of "a").

14. Do you think we now have enough information to solve the problem?

You could now perhaps solve the problem on your own. However, the resulting equations are still quite messy. So, instead, you may wish to use the following steps.

15. Find a formula for "$s(a+1) - s(a)$" in terms of "$s(a) - s(a-1)$". [**Hint:** Start with your answer to question 13, then make the substitution $s(a) = \frac{1}{3}s(a) + \frac{2}{3}s(a)$, and then re-arrange the resulting equation so that the left-hand side is just "$s(a+1) - s(a)$".]

16. Now let $x = s(1)$, an unknown quantity. In terms of x, what is $s(1) - s(0)$? What is $s(2) - s(1)$? What is $s(3) - s(2)$? What is $s(8) - s(7)$?

17. In terms of x, what is $s(2)$? What is $s(3)$? What is $s(4)$? What is $s(8)$?

18. Since we know what $s(8)$ equals, therefore what does x have to be?

19. Putting all of this together, solve for $s(a)$ for all a between 1 and 7. (If possible, try to obtain a simple, general formula for $s(a)$.)

20. Finally, what is the probability $s(6)$, that we wanted to find?