**Time to $L^2$ for certain random walks on compact Lie groups***

(Notes in progress, 1994.)

Ursula Porod and Jeffrey S. Rosenthal

On the unitary group $U(N)$, consider the random walk with step distribution given by the pushforward of the measure $C_a(\sin(\theta/2))^a d\theta \times d\lambda$ under the map $(x,\theta) \mapsto x^{-1}\mathrm{diag}(e^{i\theta},1,\ldots,1)x$, where $x \in U(N)$, $0 \le \theta < 2\pi$, $\lambda$ is normalized Haar measure on $U(N)$, and $C_a = \left(\int_0^{2\pi} (\sin(\theta/2))^a d\theta\right)^{-1}$. We take $a$ to be an integer between 0 and $N-1$.

Let $\mu_k$ be the distribution of this random walk after $k$ steps (where the starting distribution $\mu_0$ is a point mass at the identity element of $U(N)$). We are interested in the convergence of $\mu_k$ to $\lambda$.

The case $a = N-1$ was studied in Porod's thesis, and it was proved that $\|\mu_k - \lambda\|_{L^2(\lambda)} \le Ae^{-Bc}$ when $k = \frac{1}{2}N\log N + cN$, where $A$ and $B$ are positive constants. Then since $\|\mu_k - \lambda\|_{T.V.} \le \frac{1}{2}\|\mu_k - \lambda\|_{L^2(\lambda)}$, her results implied convergence rates (in fact a cut-off phenomenon!) in total variation distance.

We have now observed that when $a = N-1$, $\mu_k$ in fact has a density in $L^2(\lambda)$ for $k \ge O(N)$.

Porod also showed that when $a = 0$, $\|\mu_k - \lambda\|_{L^2(\lambda)}$ was infinite for $k < \frac{1}{2}(N^2 - N) + 1$. This dramatically different behaviour prompted the present study, whose goal is to understand the $L^2$ convergence for intermediate values of $a$. In particular, we are interested in conditions on $k$ as a function of $a$ and $N$ which would guarantee that $\mu_k$ is in $L^2(\lambda)$.

Using Fourier analysis and computing characters in a manner similar to the computation in Porod's thesis, we have shown that

* Dedicated to the memory of Onion Duck.

$$\|\mu_k - \lambda\|_{L^2(\lambda)} \;=\; K_{a,N,k} \sum_{\lambda_1<\lambda_2<\ldots<\lambda_N} \left( \sum_{\substack{j=1 \\ \lambda_j \le a}}^{N} (-1)^j \frac{\binom{-\lambda_j - a + N - 2}{N-2-2a}}{\prod\limits_{r=j+1}^{N} (\lambda_r - \lambda_j) \prod\limits_{r=1}^{j-1} (\lambda_j - \lambda_r)} \right)^{2k}$$

$$\times \left( \prod_{1 \le r < s \le N} (\lambda_s - \lambda_r) \right)^2 - 1 \,,$$

where $K_{a,N,k}$ is an explicit constant depending on $a$, $N$ and $k$. Here the sum is taken over all $N$-tuples of (positive or negative) integers $(\lambda_1, \lambda_2, \ldots, \lambda_N)$ satisfying $\lambda_1 < \lambda_2 < \ldots < \lambda_N$, and $\binom{-\lambda_j - a + N - 2}{N-2-2a}$ is a binomial coefficient.

We have further shown (by considering the sum over $m$ of terms with $(\lambda_1, \ldots, \lambda_N) = (-m, m, 2m, \ldots, (N-1)m)$) that this sum is infinite for $k < (N^2 - N + 1)/2(a+1)$. (That is, $\mu_k$ is not a measure in $L^2(\lambda)$ for this range of $k$.)

(Andrey Feuerverger has now obtained similar lower bounds by related methods.)

On the other hand, a remark by Gerard Letac made us realize that, since the above measures are mutually absolutely continuous for different values of $a$, therefore since for $a = N - 1$ the measure $\mu_k$ is absolutely continuous with respect to $\lambda$ for $k \ge O(N)$, therefore this same is true for any value of $a$.

The difficulties with further estimating the sum are that in the inside alternating sum, the individual terms may be going to infinite for large values of the $\lambda_j$, even though we know that the total value of the alternating sum is bounded by a constant. This makes analysis of the sum extremely sensitive.

One idea we had was to use spherical coordinates for $(\lambda_1, \ldots, \lambda_N)$, and to approximate the sum by an integral. Since we were only interested in the *finiteness* of the sum, we need only consider those $(\lambda_1, \ldots, \lambda_n)$ sufficiently far from the origin.