STA 2111 (Graduate Probability I), Fall 2024

Homework #1 Assignment: worth 10% of final course grade.

Due: in class by 2:10 p.m. **sharp** (Toronto time) on Thursday Sept 26.

INITIAL REQUEST:

- **Right away** (before Sept. 19), please **email** j.rosenthal@math.toronto.edu with a simple **"headshot" photo** (medium-resolution is fine), and specify **where you sit in class** from the student's perspective (e.g. "fourth row from the front, towards the left"), to help me keep track of who is who. Please also include your full name and usual nickname (if you have one) and student number and department and program and year.

GENERAL NOTES:

- Homework assignments are to be solved by each student individually. You may discuss questions in general terms with other students, and look up general topics in books and internet. But you must solve the problems on your own, and do all of your own writing.

- You should provide very complete solutions, **EXPLAINING ALL REASONING** very clearly. Please submit your assignment as hard copy at the beginning of class. Try to make your homework neat and easy to read, e.g. typeset in tex/latex, or printed clearly.

- **Late penalty:** 1–5 minutes late is -5%; 5–15 minutes late is -10%; otherwise if x days late then $-20\% \times \text{ceiling}(x)$. So, please don't be late!

THE ACTUAL ASSIGNMENT:

1. Let $\Omega = \{1,2,3,4\}$, and let $\mathcal{J} = \Big\{\emptyset, \{1\}, \{2\}, \{3,4\}, \Omega\Big\}$. Define $\mathbf{P} : \mathcal{J} \to [0,1]$ by $\mathbf{P}(\emptyset) = 0$, $\mathbf{P}\{1\} = 1/7$, $\mathbf{P}\{2\} = 2/7$, $\mathbf{P}\{3,4\} = 4/7$, and $\mathbf{P}(\Omega) = 1$.

(a) [**3**] Prove that $\mathcal{J}$ is a semi-algebra.

(b) [**4**] Find $\mathbf{P}^*(A)$ and $\mathbf{P}^*(A^C)$, where $A = \{2,3\} \subseteq \Omega$ and $\mathbf{P}^*$ is outer measure.

(c) [**4**] Determine whether or not $A \in \mathcal{M}$, where $\mathcal{M}$ is the σ-algebra constructed in the proof of the Extension Theorem. [Hint: Perhaps consider the case $E = \Omega$.]

2. [**5**] Prove that the extension $(\Omega, \mathcal{M}, \mathbf{P}^*)$ constructed in the proof of the Extension Theorem must be "complete", i.e. if $A \in \mathcal{M}$ with $\mathbf{P}^*(A) = 0$, and $B \subseteq A$, then $B \in \mathcal{M}$.

3. Let $\Omega = \{1,2,3,4\}$, and $\mathcal{F} = 2^\Omega$ the collection of all subsets of Ω, and

$$\mathcal{C} \;=\; \Big\{\emptyset, \{1,2\}, \{2,3\}, \{3,4\}, \Omega\Big\}\,.$$

Define functions $\mu, \nu : \mathcal{F} \to [0,1]$ by $\mu(A) = \frac{1}{2}\mathbf{1}_A(1) + \frac{1}{2}\mathbf{1}_A(3)$ and $\nu(A) = \frac{1}{2}\mathbf{1}_A(2) + \frac{1}{2}\mathbf{1}_A(4)$, where e.g. $\mathbf{1}_A(3) = 1$ if $3 \in A$ or $\mathbf{1}_A(3) = 0$ if $3 \notin A$.

(a) [**5**] Prove that $(\Omega, \mathcal{F}, \mu)$ is a valid probability triple. (It then follows similarly that $(\Omega, \mathcal{F}, \nu)$ is also a valid probability triple.)

(b) **[3]** Determine whether $\mathcal{C}$ is an algebra.

(c) **[3]** Determine whether $\mathcal{C}$ is a semi-algebra.

(d) **[5]** Find $\sigma(\mathcal{C})$, the smallest σ-algebra containing all elements of $\mathcal{C}$.

(e) **[3]** Determine whether $\mu(A) = \nu(A)$ for all $A \in \mathcal{C}$.

(f) **[3]** Determine whether $\mu(A) = \nu(A)$ for all $A \in \mathcal{F}$.

(g) **[3]** Explain why these facts do not contradict our theorem about uniqueness of extensions of probability measures.

4. For any interval $I \subseteq [0,1]$, let $\mathbf{P}(I)$ be the <u>length</u> of I.

(a) **[5]** Prove that if $I_1, I_2, \ldots, I_n$ is a <u>finite</u> collection of intervals, and if $\bigcup_{j=1}^{n} I_j \supseteq I_*$ for some interval I_*, then $\sum_{j=1}^{n} \mathbf{P}(I_j) \geq \mathbf{P}(I_*)$. [Hint: Suppose I_j has left endpoint a_j and right endpoint b_j, and first re-order the intervals so $a_1 \leq a_2 \leq \ldots \leq a_n$.]

(b) **[5]** Prove that if $I_1, I_2, \ldots$ is a countable collection of <u>open</u> intervals, and if $\bigcup_{j=1}^{\infty} I_j \supseteq I_*$ for some <u>closed</u> interval I_*, then $\sum_{j=1}^{\infty} \mathbf{P}(I_j) \geq \mathbf{P}(I_*)$. [Hint: You may use the <u>Heine-Borel Theorem</u>, which says that if a collection of open intervals contain a closed interval, then some <u>finite sub-collection</u> of the open intervals also contains the closed interval.]

(c) **[5]** Prove that if $I_1, I_2, \ldots$ is *any* countable collection of intervals, and if $\bigcup_{j=1}^{\infty} I_j \supseteq I_*$ for *any* interval I_*, then $\sum_{j=1}^{\infty} \mathbf{P}(I_j) \geq \mathbf{P}(I_*)$. [Hint: Extend the interval I_j by $\epsilon\, 2^{-j}$ at each end, and decrease I_* by ϵ at each end, while making I_j open and I_* closed. Then use part (b).] (Note: This is the "countable monotonicity" property needed to apply the Extension Theorem for the Uniform[0,1] distribution, to guarantee that $\mathbf{P}^*(I) \geq \mathbf{P}(I)$.)

(d) **[4]** Suppose we instead defined $\mathbf{P}(I)$ to be the <u>square</u> of the length of I. Show that in that case, the conclusion of part (c) would <u>not</u> hold.

[END; total points = 60]